Owari I. Marching Groups and Periodical Queues

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Abstract

Owari is a board game, originally played in Africa and Asia, made of holes arranged in a circular way and containing pebbles. A typical move is to scoop a hole and to sow its pebbles one by one into the subsequent holes. A marching group is made of n successive holes whose numbers of pebbles are given by the sequence $[n, n-1, \ldots, 2, 1]$. It is invariant by the sowing transformation that scoops the hole with n pebbles. We study the distributions of pebbles that reappear periodically when repeted sowing transformations are applied.

1 Owaris

A *closed owari* is made of holes placed along an oriented closed curve and containing pebbles. To *sow from a hole h* is to scoop the pebbles in h and to sow them, one by one, into the subsequent holes.

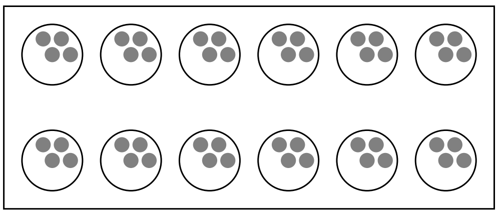

Figure 1: Awele. This closed owari, made of 12 holes containing 4 pebbles placed along two opposite sides of a rectangle, is the initial configuration of the awele game.

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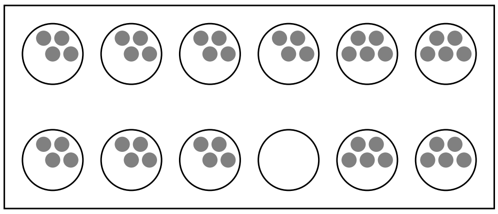

Figure 2: Sowing. Here we have sown from the fourth hole in the bottom side of the awele.

If we sow from h and the number of holes is lower than the number of pebbles in h then the sowing process *overlaps* in the sense that a pebble falls in h, so that no empty hole exists in the new distribution of pebbles. Overlappings can be avoided by considering open owaris.

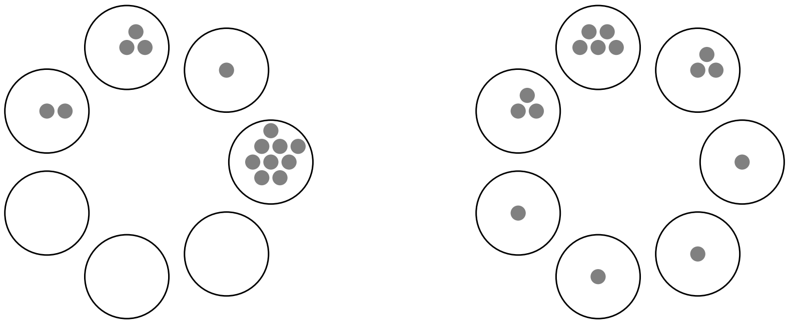

Figure 3: Overlapping. These closed owaris have 7 holes placed along a circle with the counterclockwise sense of rotation. The second owari is derived from the first one by sowing from the hole h with 9 pebbles. The sowing process overlaps in the sense that a pebble falls in h; there are no longer empty holes.

An *open owari* is made of holes placed along an oriented open line, from left to right by convention, and of a finite number of pebbles distributed into the holes. It is also required that every hole has a hole on its left and a hole on its right. Sowing from a hole h is defined like in a closed owari. Overlappings can no longer occur because there are infinitely many holes after h.

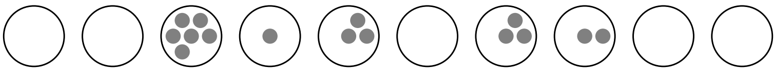

Figure 4: Open owari. This open owari is made of holes placed along a straight line oriented from left to right. We may think to an open owari as a limit of a closed owari when the number of holes increases infinitely. There is an infinite set of holes on the right and on the left. The leftmost nonempty hole, which contains 6 pebbles, is the tail. The rightmost nonempty hole, which contains 2 pebbles, is the head.

An *owari* is either a closed owari or an open owari. For every integer $i \geq 0$ and every hole h we will denote by $h + i$ the hole that is i position after h;

in particular $h + 0 = h$. Let $w(h)$ be the number of pebbles in hole h. The mapping $h \mapsto w(h)$ is the *weight function*. Obviously $w(h) \geq 0$. We will give a meaning in Section 5 to the case where $w(h) < 0$. An owari is *empty* if its holes are empty. The leftmost (rightmost) nonempty hole of a nonempty open owari is the *tail* (*head*) of that open owari.

2 Queues

A *partition* of a positive integer w is a sequence of positive integers

$$P = [w_0, w_1, \ldots, w_{l-1}]$$

such that

$$w = \sum_{0 \leq i < l} w_i.$$

We refer the reader to the wiki on integer compositions and partitions on the web. It is known that the number of partitions of w is equal to 2^w.

A *queue* is a nonempty open owari that has no empty holes between its tail and its head. We construct a queue Q that *represents* P by considering an empty open owari, a hole h of that owary and by placing w_i pebbles in hole $h + i$, for $0 \leq i < l$. We also say that P is the *weight sequence* of Q.

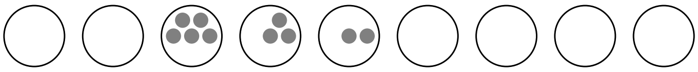

Figure 5: Queue. This open owari has no empty holes between its tail and its head. It is a queue that represents the partition $[5, 3, 2]$ of the integer 10.

We will briefly say that we *derive* a queue Q to mean that we sow Q from its tail. Then we get a new queue Q' that represents a partition P' of the integer w. The partition P' is also said to be derived from P.

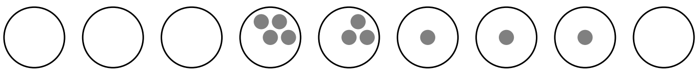

Figure 6: Derivation. This queue is obtained by deriving (sowing from the tail) the preceding queue.

Derivation maps the set of partitions of the same integer w into itself. Since this set is finite, when we start with a given partition and we iterate derivation we end with a periodical cycle of partitions. More precisely a partition of an integer (a queue) is *periodical* if, for some integer $p > 0$, we obtain the same partition (the same queue up to a translation) after p derivations. The smallest value of p is the *period* of the partition (queue).

A queue of period 1 is a *marching group*. Ron Eglash points out the importance of marching groups in actual owari games [1].

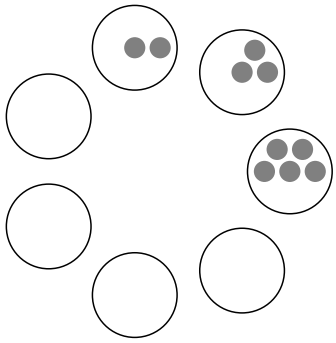

Figure 7: Queue as a closed owari. This closed owari also represents the partition $[5, 3, 2]$ of the integer 10. When repeatedly deriving this queue we are not sure that no overlapping will occur. It is easier to study repeated derivations in the frame of open owaris.

$$[2, 1, 1] \to [2, 2] \to [3, 1] \to [2, 1, 1]$$

Table 1: A partition of the integer 4 with a period equal to 3.

3 Augmented marching groups

It is easy to verify that a marching group has a weight sequence of the form $[n, n-1, \ldots, 2, 1]$. The integer n is the *order* of the marching group. Let us consider an integer $0 < a \leq n+1$. To *augment* the marching group by a pebbles is to add one pebble to a holes chosen among the nonempty holes and the empty hole that follows the last nonempty hole. Formally the augmented marching group is a queue with a weight sequence of the form

$$[n + a_n, n - 1 + a_{n-1}, \ldots, 2 + a_2, 1 + a_1, \widehat{0 + a_0}]$$

where each a_i is equal to 0 or 1,

$$a = \sum_{0 \leq i \leq n} a_i,$$

and the hat above $0 + a_0$ means that the term is present at the end of the partition when it is nonnull, missing otherwise. For example we can rewrite the first three weight sequences of Table 1 as

$$[2 + 0, 1 + 0, 0 + 1] \to [2 + 0, 1 + 1, \widehat{0 + 0}] \to [2 + 1, 1 + 0, \widehat{0 + 0}],$$

$$[5, 3, 2] \to [4, 3, 1, 1, 1] \to [4, 2, 2, 2] \to [3, 3, 3, 1] \to [4, 4, 2] \to [5, 3, 1, 1] \to$$
$$[4, 2, 2, 1, 1] \to [3, 3, 2, 2] \to [4, 3, 3] \to [4, 4, 1, 1] \to [5, 2, 2, 1] \to [3, 3, 2, 1, 1] \to$$
$$[4, 3, 2, 1] \to [4, 3, 2, 1]$$

Table 2: Repeated derivations. By repeatedly deriving the partition $[5, 3, 2]$ of the integer 10 we arrive at the partition $[4, 3, 2, 1]$, which is invariant by any further derivation. This partition is the weight sequence of the marching group of order 4. This example is given by Ron Eglash [1].

which makes apparent the underlying marching group $[2,1]$ augmented by one pebble. We will prove the following results.

Theorem 1 *A queue is periodical if and only if it is an augmented marching group.*

Theorem 2 *An integer p is the period of a marching group of order n augmented by a pebbles if and only if one of the following conditions holds.*

- $a = n + 1$ *and* $p = 1$.
- $1 \leq a \leq n$ *and* $p = \frac{n+1}{d}$, *where d is a divisor of $n + 1$ and a.*

w	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
p	1	2	1	3	3	1	4	4, 2	4	1	5	5	5	5	1

Table 3: Realizable period p for a periodical queue with w pebbles. This table is derived from Theorem 2. It appears in [2] but the period 2 for a queue with 8 pebbles is missing. A realization of that period is $[4,2,2] \to [3,3,1,1] \to [4,2,2]$.

Theorem 2 and the sufficient condition of Theorem 1 are easy to prove. This is done in Section 4. To prove that a periodical queue O_2 is an augmented marching group we will consider any marching group O_1 defined on the same set of holes as O_2 and with the same tail. We will decompose the sowing operation from the tail of O_2 by first sowing the pebbles in the tail of O_1, then by sowing the pebbles in $O_2 - O_1$. The difference $O_2 - O_1$ is the owari defined by the weight function $w_2 - w_1$, where w_2 and w_1 are the weight functions of O_2 and O_1, respectively. If $w_2 - w_1$ takes only positive or null values it is actually the weight function of an owari. In Section 5 we define signed owaris and their sowing operations in order to give a meaning to the owari $O_2 - O_1$ when $w_2 - w_1$ takes negative values. In Section 6 we define the amalgamation of O_1 and O_2, which allows to specify how the sowing operation from the tail of O_2 can be decomposed into the sowing operation from the tail of O_1 completed by a sowing operation in the signed owari $O_2 - O_1$. Then we prove the necessary condition of Theorem 1.

4 First proofs

Proof of Theorem 1, sufficient condition. Let us consider the weight sequence of an augmented marching group of order n,

(1) $\quad [n + a_n, n - 1 + a_{n-1}, \ldots, 2 + a_2, 1 + a_1, \widehat{0 + a_0}]$,

and the binary sequence

(2) $\quad [a_n, a_{n-1}, \ldots, a_2, a_1, a_0]$,

which we call the *augmentation sequence.*

After scooping the $n+a_n$ pebbles in the tail of the marching group the weight sequence of the queue formed by the remaining nonempty holes becomes

$$(3) \quad [n-1+a_{n-1}, n-2+a_{n-2}, \ldots, 2+a_2, 1+a_1, \widehat{0+a_0}].$$

By sowing the $n+a_n$ pebbles, we first increase by 1 the n terms of (3), and we eventually add a_n pebbles to the end. This yields

$$[n+a_{n-1}, n-1+a_{n-2}, \ldots, 3+a_2, 2+a_1, 1+a_0, \widehat{0+a_n}],$$

which is still the weight sequence of an augmented marching group. The new augmentation sequence is

$$[a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0, a_n],$$

which is derived from (2) by a left rotation. Since (2) has length $n+1$ it will be reproduced after $n+1$ sowing transformations. Accordingly the augmented marching group will be reproduced after $n+1$ sowing transformations. □

Proof of Theorem 2. We still denote by (1) the weight sequence of the augmented marching group.

Case 1. $a = n+1$. The augmentation sequence is made of 1's and so it is reproduced after any rotation. Therefore the period is equal to 1. This is consistent with the fact that the augmentation yields the marching group of order $n+1$.

Case 2. $1 \leq a \leq n$. Let d be a divisor of $n+1$ and a. Let $p = \frac{n+1}{d}$, $x = \frac{a}{d}$ and $y = p - x$. Let us consider the binary sequence

$$(4) \quad [1^x, 0^y, 1^x, 0^y, \ldots, 1^x, 0^y],$$

where 1^x stands for 1 repeted x times, 0^y stands for 0 repeted y times and the subsequence $1^x, 0^y$ is repeted d times. Sequence (4) has length $d(x+y) = n+1$. Let us assume that the augmentation sequence (2) is equal to (4). Since the subsequence $1^x, 0^y$ has length p, (4) is invariant after a rotation by p positions, and so (1) is reproduced after p sowing transformations. It cannot be reproduced after a number of sowing transformations $q < p$, otherwise (4) would be invariant after a rotation by q positions, which is impossible because $y > 0$. Therefore (1) has period p. □

5 Signed owaris

An owari is *signed* if each pebble is provided with a *weight* equal to $+1$ or -1. In order to define how to *sow from a hole h* we think to a hand moving above the holes and sowing the pebbles one by one. At the begining the hand contains the pebbles scooped in h and is above h. Then we proceed according to the following algorithm.

1. Select a pebble in the hand.

2. If the selected pebble has weight +1, then move the hand one hole forward and sow the pebble into the hole below the hand.

3. If the selected pebble has weight -1, then sow the pebble into the hole below the hand and move the hand one hole backward.

4. Return to Step 1 until the hand is empty.

We point out that in steps 2 and 3 the actions of sowing and moving the hand are done in reverse orders.

The result of the sowing operation depends on the order to select the pebbles except if all the pebbles in h have the same weight. If all the pebbles have weight +1 then Rule 2 implies that they are sown, one by one, in the holes that follow h. We retrieve the sowing operation as defined in Section 1. Accordingly we identify an owari with a signed owari whose pebbles have weight +1. If all the pebbles have weight -1 then Rule 3 implies that they are sown, one by one, in h and the holes that precede h.

We now define an equivalence relation between owaris that is respected by the sowing operation, independantly of the selection order of the pebbles. To *reduce* a hole is to remove pairs of pebbles with opposite weights from that hole until there remains only pebbles with the same weight. To *reduce* a signed owari is to reduce every hole of that owari. Two signed owaris are *equivalent* if they have the same reduction.

Lemma 1 *Let h be a hole in a signed owari O. Let O' be the signed owari obtained from O by reducing h. Every signed owari obtained by sowing O from h is equivalent to the signed owari obtained by sowing O' from h.*

Proof. Let p and n be the respective numbers of positive pebbles and negative pebbles in h. Let us consider the *selection sequence* $s = [s_1, s_2, \ldots, s_{p+n}]$, where s_i is the weight of the i^{th} selected pebble when sowing from h, $1 \leq i \leq p + n$. Unless h is reduced, and in this case there is nothing to prove, there is an index i such that $1 \leq i < p + n$ and $s_i + s_{i+1} = 0$. Let h' be the hole under the hand just when selecting the i^{th} pebble.

Case 1. $s_i = +1$. According to Rule 2 the hand moves forward to a new hole h'' and drops the pebble into h''. The next selected pebble has weight -1. According to Rule 3 the hand drops that pebble into h'' and it moves backward to h'.

Case 2. $s_i = -1$. According to Rule 3 the hand drops the pebble into h' and moves backward to a new hole h'''. The next selected pebble has weight +1. According to Rule 2 the hand moves forward to h' and drops that pebble into h'.

In both cases two pebbles of opposite weights have been dropped into some hole (h' or h'') and the hand is still above h'. This is like we delete first two pebbles of opposite weights from h, we drop them into some hole, and we sow from h according to the selection sequence s less the terms s_i and s_{i+1}. We can go on this way by successively deleting from h some pairs of pebbles with opposite weights until h is reduced, dropping one by one the pairs of pebbles into some holes, then sowing h when it is reduced. Finally by removing the pairs of pebbles with opposite weights that have been dropped, we obtain an equivalent signed owari. $\square$

6 Aggregates of two queues

The *weight function* of a signed owari is the function $w : H \to \mathbf{Z}$, where H is the set of holes and $w(h)$ is the sum of the weights of the pebbles in hole h. We note that two signed owaris are equivalent iff they have the same weight function. We also note that if an empty owari on the set of holes H is given with a mapping $w : H \to \mathbf{Z}$, then we construct a reduced signed owari with weight function w by putting $|w(h)|$ pebbles in each hole h with a weight equal to +1 if $w(h) \geq 0$, equal to -1 otherwise.

Let us consider two queues O_1 and O_2 on the same set of holes H and the same tail h. We transform O_1 and O_2 into signed owaris by defining a weight +1 for each pebble. Let w_1 and w_2 be the weight functions of O_1 and O_2, respectively. We denote by $O_2 - O_1$ the reduced signed owari defined on the set of holes H by the weight function $w_2 - w_1$. The *aggregate* of O_1 and O_2 is the signed owari defined on the set of holes H as follows.

- Each pebble is colored green or red.
- The signed owari constituted of the only green pebbles is equal to O_1.
- The signed owari constituted of the only red pebbles is equal to $O_2 - O_1$.

The signed owari constituted of all the pebbles, green or red, is equivalent to O_2 because the weight of a hole x is equal to the sum of $w_1(x)$, for the green part, and $w_2(x) - w_1(x)$, for the red part, which is equal to $w_2(x)$.

To *derive the aggregate* is to sow from h the signed owari defined by all the pebbles, green or red, according to the algorithm defined in Section 5. However we begin by sowing the green pebbles, then we complete by sowing the red pebbles. During the sowing operation we also remove any pair of red pebbles with opposite weights in order to reduce the signed owari restricted to the red pebbles. When we sow the green pebbles we sow from h in O_1 and when we complete the sowing operation we sow from h in O_2. Thus we get the aggregate of the signed owaris O'_1 and O'_2 obtained by sowing O_1 and O_2 from h. We point out that O'_1 is the queue derived from O_1 and that O'_2 is a signed owari equivalent to the queue derived from O_2.

Lemma 2 *If we construct the aggregate of a marching group and a periodical queue with the same tail, then the red pebbles in the aggregate have the same weight.*

Proof. Let us denote by O_1 the marching group, by n the order of this marching group, by O_2 the periodical queue, by h the common tail of O_1 and O_2 and by A their aggregate. For every integer $i \geq 0$ let us denote by A^i the aggregate obtained by deriving i times A.

Claim 1. No pair of red pebbles with opposite weights is removed during the derivation of any aggregate A^i. If that occured then, for every integer $m > i$, there would be less pebbles in A^m than in A. However, by taking for m a multiple of the period of O_2, we retrieve aggregate A translated by m holes, whose number of pebbles equals the number of pebbles of A, a contradiction.

For every pair of integers $i, j \geq 0$ let $r^i(j)$ be the weight of the red pebbles of aggregate A^i in hole $h + j$.

Claim 2. If $j \geq i$ and $r^i(j) \neq 0$ then $r^j(j) \neq 0$ and $r^j(j)$ has the same sign as $r^i(j)$. The property is obvious if $i = j$. Let us assume now that $j > i$. When deriving A^i no pair of red pebbles with opposite weights is removed from $h + j$ by Claim 1. Therefore $r^{i+1}(j) \neq 0$ and $r^i(j)$ has the same sign as $r^{i+1}(j)$. After performing $j - i$ derivations we obtain the desired property.

Claim 3. If $r^j(j) < 0$ then $r^{j+n}(j + n) < 0$. Indeed by deriving A^j we first sow the n green pebbles of weight +1, then we sow the red pebbles of weight -1. The first of these red pebbles falls into hole $h + j + n$ by Step 3 of the sowing algorithm in Section 5. Accordingly $r^{j+1}(j + n) < 0$. This implies $r^{j+n}(j + n) < 0$ by Claim 2.

Claim 4. If $r^j(j) > 0$ then $r^{j+n+1}(j + n + 1) > 0$. Indeed by deriving A^j we first sow the n green pebbles of weight -1, then we sow the red pebbles of weight +1. The first of these red pebbles falls into hole $h + j + n + 1$ by Step 2 of the sowing algorithm in Section 5. Accordingly $r^{j+1}(j + n + 1) < 0$. This implies $r^{j+n+1}(j + n + 1) < 0$ by Claim 2.

Let us now suppose, for a contradiction, that we can find in A a red pebble with weight +1 in hole $h + x$ and a red pebble with weight -1 in hole $h + y$. We may assume $x = 0$ and $y > 0$ by possibly replacing A by A^x. Thus we have $r^x(x) > 0$ and, according to Claim 4,

$$(5) \qquad r^{x+k(n+1)}(x + k(n + 1)) > 0$$

for every integer $k \geq 0$. According to Claim 2 applied with $i = 0$ and $j = y$ we have $r^y(y) > 0$ and, according to Claim 3, $r^{y+kn}(y + kn) < 0$ for every integer $k \geq 0$. By setting $k = y$, we get $r^{k+kn}(k + kn) < 0$, which contradicts (5). □

Proof of Theorem 1, necessary condition. Let O_2 be a periodical queue, let N be the number of pebbles in O_2, let n be the maximal integer such that

$$(6) \qquad \frac{n(n + 1)}{2} \leq N$$

and let n' be the minimal integer such that

$$(7) \qquad \frac{n'(n' + 1)}{2} > N.$$

Obviously $n' = n + 1$. Let O_1 and O'_1 be the marching groups of orders n and n', respectively, with the same set of holes and the same tail as O_2. Let w_2, w_1 and w'_1 be the weight functions of O_2, O_1 and O'_1, respectively.

By Lemma 2 the red pebbles of the aggregate of O_1 and O_2 have the same weight. This weight is equal to -1 because O_1 has no more pebbles than O_2 by

(6). Since the weight function of the owari defined by the red pebbles is equal to $w_2 - w_1$ and O_1 is a marching group of order n, we have

$$w_2(h+i) \geq w_1(h+i) = n - i$$

for $0 \leq i \leq n$. Similarly the red pebbles of the aggregate of O_1' and O_2 have the same weight +1 by (7). This implies

$$w_2(h+i) \leq w_1'(h+i) = n' - i = n + 1 - i$$

for $0 \leq i \leq n$. Therefore

$$n - i + 1 \geq w_2(h+i) \geq n - i,$$

and O_2 is derived from the marching group of order n by adding one pebble to each hole $h+i$ such that $w_2(h+i) = n - i + 1$. □

References

[1] Ron Eglash, L'algorithmique ethnique, Pour la Science, Dossier N° 47, avril/juin 2005.

[2] Ron Eglash, African Fractals: modern computing and indigenous design. New Brunswick: Rutgers University Press 1999.