MATH-2400 **FINAL EXAM**
INTRO DIFFERENTIAL EQUATIOS **Fall 2001**

Please answer all **7** questions, showing your work in detail. **One sheet of notes is permitted for the test.**

1.(12p.) (a) Solve the initial value problem

$$y' + \frac{2}{t}y = \frac{\cos t}{t^2} + 1, \quad y(\pi) = 0.$$

(b) Is the equation linear or nonlinear?
(c) On what interval is the solution valid

2.(a)(6p.) The plot below shows graphs of solutions of an equation of the form $dy/dt = f(y)$. (a) What is the nature of solutions $y = 1$ and $y = 3$? (b) Give an example of $f(y)$ so that solutions of $dy/dt = f(y)$ have the graphs such as those shown.

2.(b)(9p.) Find an implicit algebraic equation for the solution of the problem,

$$\frac{dy}{dx} = \frac{e^x}{y+1}, \quad y(0) = 0.$$

3. (a)(7p.) Find the general (real) solution of the equation

$$u'' - 2u' + 2u = 0.$$

(b)(8p.) Find a particular solution of

$$u'' - 2u' + 2u = 2 + 2t$$

4. Suppose $f(x)$ has a Fourier series,

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{3})$$

where

$$f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 3. \end{cases}$$

(a)(8p.) Find the coefficients c_n of the series above.

(b)(5p.) Denote the sum of the series in part (a) by f. Use properties of the series (periodic, even or odd) to sketch its graph and determine to which values the series converges at the following points

$f(1/2) =$, $f(-1/2) =$, $f(1) =$, $f(4) =$, $f(-2) =$, $f(-6) =$

5.(a)(6p.) Find the eigenvalues and eigenvectors of the matrix $C = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

(b)(7p.) $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ has $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as eigenvectors with eigenvalues -3 and -1 correspondingly. Find the solution of the initial value problem $\mathbf{x}' = A\mathbf{x}, \mathbf{x}(0) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

(c)(7p.) Sketch a phase portrait of the system $\mathbf{x}' = A\mathbf{x}$ of part (b).

6. Consider the nonlinear system

$$x' = (-1 + y)x - \frac{1}{2}x, \quad y' = (1 - x)y - \frac{1}{2}y.$$

(a)(5p.) Find all the equilibrium solutions (critical points of the system).

(b)(5p.) Find the linear system that approximates the given system in the neighborhood of $(0, 0)$, and determine the type and stability of $(0, 0)$.

7. Consider the heat conduction problem

$$u_t = u_{xx}, \quad 0 < x < 10, \ t > 0,$$

$$u_x(0, t) = 10, \quad t > 0$$

$$u(10, t) = 50, \quad t > 0$$

$$u(x, 0) = 40 - 4x, \quad 0 < x < 10$$

(a)(7p.) Find the steady state temperature distribution $v(x)$ that will be approached as $t \to \infty$.
(b)(3p.) Let $u(x, t) = v(x) + w(x, t)$, where v is the steady solution from part (a), i.e., let w be the transient temperature distribution. **Write down** the problem satisfied by w (i.e., the equation, the initial and boundary conditions). **Do not solve** the problem for w you obtain in this question.

(d)(5p.) Use the method of separation of variables to replace $u_{xx} + xu_t = 0$ by a pair of ordinary differential equations.