Math-2400 LINEAR ALGEBRA

Systems of Linear Algebraic Equations. A set of n linear algebraic equation with n unknowns,

$$\begin{aligned} a_{11}x_1 + ... + a_{1n}x_n &= b_1 \\ &\dots \\ a_{n1}x_1 + ... + a_{nn}x_n &= b_n \end{aligned}$$

can be written in the matrix form: $\mathbf{Ax} = \mathbf{b}$. The system is *homogeneous* if $\mathbf{b} = \mathbf{0}$; otherwise, it is *nonhomogeneous.*
If $\det \mathbf{A}$ is not 0 (i.e., if $\mathbf{A}$ is nonsingular and therefore $\mathbf{A}^{-1}$ exists), then the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

In particular, the homogeneous problem $\mathbf{Ax} = \mathbf{0}$, corresponding to $\mathbf{b} = \mathbf{0}$, has only the trivial solution $\mathbf{x} = \mathbf{0}$.
If $\det \mathbf{A} = 0$, then the homogeneous system $\mathbf{Ax} = 0$ has infinitely many nonzero solutions. On the other hand, the nonhomogeneous system $\mathbf{Ax} = \mathbf{b}$ in general does not have any solutions, unless the vector $\mathbf{b}$ satisfies certain extra conditions. Namely the dot product

$$(\mathbf{b}, \mathbf{y}) = 0$$

for all vectors $\mathbf{y}$ that solve the system $\mathbf{A}^*\mathbf{y} = \mathbf{0}$. If this condition is satisfied then the system has infinitely many solutions.
Eigenvalues and eigenvectors. A vector $\mathbf{x}$ is called an **eigenvector** of the matrix $\mathbf{A}$, if it has the property that $\mathbf{Ax}$ is proportional to $\mathbf{x}$, i.e. that $\mathbf{Ax} = \boldsymbol{\lambda}\mathbf{x}$ where $\boldsymbol{\lambda}$ is a scalar. This equation can be put in the form of a homogeneous system

$$(\mathbf{A} - \boldsymbol{\lambda}\mathbf{I})\mathbf{x} = \mathbf{0},$$

which has nonzero solutions $\mathbf{x}$ if and only if

$$\det(\mathbf{A} - \boldsymbol{\lambda}\mathbf{I}) = 0.$$

Values of $\boldsymbol{\lambda}$ that satisfy this equation are called eigenvalues of the matrix $\mathbf{A}$.